Derivation of E=mc^2

To derive a relativistic expression for kinetic energy, we simply apply the Work-Kinetic Energy Theorem and calculate the work done by a force on an object:

$$K = \int F \cdot dx = \int \frac{dp}{dt} \cdot dx = \int v \cdot dp$$

We must use the relativistic expression for momentum, p=γmv, and integrate from some initial condition (lets say speed = 0) to some final condition, (let's say speed v_f.) The above expression becomes:

$$K = \int_0^{v_f} v \cdot d\left(\frac{mv}{\sqrt{1 - v^2/c^2}} \right)$$

One way to solve this is with integration by parts:

$$K = v\left(\frac{mv}{\sqrt{1 - v^2/c^2}} \right) - \int_0^{v_f} \frac{mv}{\sqrt{1 - v^2/c^2}}\, dv = \frac{mv^2}{\sqrt{1 - v^2/c^2}} - m\int_0^{v_f} \frac{vdv}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{mv^2}{\sqrt{1 - v^2/c^2}} + mc^2\sqrt{1 - v^2/c^2}\,\Bigg|_0^{v_f}$$

$$= \left(\frac{mv_f^{\,2}}{\sqrt{1 - v_f^{\,2}/c^2}} + mc^2\sqrt{1 - v_f^{\,2}/c^2} \right) - \left(0 + mc^2\right)$$

This further simplifies as:

$$K = \frac{m}{\sqrt{1 - v_f^2/c^2}}\left(v_f^2 + c^2\left(1 - v_f^2/c^2 \right) \right) - mc^2$$

$$= \frac{m}{\sqrt{1 - v_f^2/c^2}}\left(v_f^2 + c^2 - v_f^2 \right) - mc^2$$

So that we get:

$$K = \frac{mc^2}{\sqrt{1 - v_f^2/c^2}} - mc^2$$

$$K = \gamma mc^2 - mc^2$$

Where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

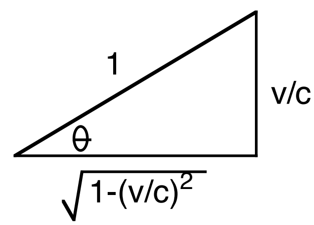
Derivation of E=mc²

Notice that the kinetic energy depends on two terms, one of which depends on the speed of the object, the other of which is a constant for the object. Oftentimes this is rewritten as $\gamma mc^2 = K + mc^2$ The first term, γmc^2, is interpreted as the total energy of the object, called E. The second term, mc^2, is then interpreted as the "rest" energy of the object, and is labeled E_0. If an object were not moving, v=0 and then the energy would simply be the rest energy, or $E_0 = mc^2$. To summarize these ideas and terms: E is the total energy of an object and E_0 is the rest energy of an object where $E = \gamma mc^2$ and $E_0 = mc^2$ so that we can say $E = K + E_0$. Note that we could then rewrite the kinetic energy equation derived above as

$$K = (\gamma - 1)E_0 \qquad \& \qquad E_0 = mc^2 \qquad \& \qquad E = \gamma mc^2$$

Trig Substitution

Here is the integral from earlier with trig substitution (which may be easier, depending on how you like to look at things.) Based on the triangle to the right,



Let $\tan\theta = \dfrac{v/c}{\sqrt{1 - v^2/c^2}}$ and $\sin\theta = \dfrac{v}{c}$ so that the integral

$$K = \int_0^{v_f} v \cdot d\left(\frac{mv}{\sqrt{1 - v^2/c^2}}\right) = m\int_0^{v_f} v \cdot d\left(\frac{v}{\sqrt{1 - v^2/c^2}}\right)$$

becomes

$$\begin{aligned} K &= m\int_0^{v_f} (c\sin\theta) \cdot d(c\tan\theta) \\ &= mc^2 \int_0^{v_f} \sin\theta \sec^2\theta d\theta \\ &= mc^2 \int_0^{v_f} \cos^{-2}\theta \sin\theta d\theta \\ &= mc^2 \cos^{-1}\theta \Big|_0^{v_f} \end{aligned}$$

Putting this back into terms of v and c (refer to the triangle above) the result becomes

$$\begin{aligned} K &= \left.\frac{mc^2}{\sqrt{1 - v^2/c^2}}\right|_0^{v_f} \\ &= \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 \\ &= \gamma mc^2 - mc^2 \end{aligned}$$